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Theory of optical hysteresis for TE guided modes

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TE guiding structures consisting of an optically linear dielectric film embedded in dissimilar optically nonlinear unbounded media, are investigated for hysteresis properties. It is shown that many important features can be determined without a knowledge of the electric fields in the nonlinear media and that the eigenvalue equation yields a relation between the guided-wave vector and the field amplitude at the boundary. The power in the system is the important physical quantity and is carried by asymmetric modes that, in the linear limit, are neither odd nor even. This paper explores some of the limitations of such asymmetrically loaded linear dielectric films and shows that they can exhibit optical hysteresis.

Optically nonlinear waveguiding structures

Guiding structures consisting of relatively thin dielectric layers bounded by semi-infinite media have been of interest for some time. Such structures are destined to form part of the integrated optics systems of the future so it is of considerable interest to ask what will happen when one, or all, of the layers becomes optically nonlinear. The latter occurs when the refractive index becomes a function of the intensity of the wave it is carrying. So, if a dielectric layer is optically nonlinear, its dielectric constant will change from its linear value ϵ, say, to $\epsilon+\alpha|\boldsymbol{E}|^2$ because it is supporting an electric field $\boldsymbol{E}$. Optical nonlinearity in symmetrically loaded structures has recently been investigated by Akmediev (1982) in terms of the power flow, and subsequent work by Boardman & Egan (1984) has considerably elucidated this structure and has drawn attention to the fact that the asymmetrically loaded thin film is a non-trivial alternative.

If the guiding structure (see figure 1) consists of media with dielectric constants ϵ_i and nonlinear coefficients α_i then, after some manipulation, it can be shown that the general relation of the electric field amplitude E_0 at $z = 0$ to the electric field amplitude E_b at $z = d$ is the conic section

$$\alpha_3\left(\frac{\eta_3^2}{\alpha_3}-\frac{\eta_1^2}{\alpha_1}\right)^{-1}\left(E_b^2-\frac{\eta_3}{\alpha_3}\right)^2-\alpha_1\left(\frac{\eta_3^2}{\alpha_3}-\frac{\eta_1^2}{\alpha_1}\right)^{-1}\left(E_0^2-\frac{\eta_1}{\alpha_1}\right)^2 = 1, \quad \eta_i = \epsilon_2-\epsilon_i. \tag{1}$$

This equation emerges if it is assumed that E_0 and E_b are real, so that only the part of the conic section appearing in the first quadrant will be significant. Hence, for a particular value of E_0^2, there may exist two, one or no values of E_b^2, a circumstance that is entirely determined by the media parameters.

In general there are four types of conic section that are possible. Type I and type II are hyperbolae. Type I has a horizontal transverse axis and type II has a vertical transverse axis. The other two are ellipses, i.e. type III with a horizontal major axis and type IV with a vertical major axis. Only type II will be considered here, as an illustration, and this is shown in figure 2 for varying degrees of asymmetry between the upper and lower medium.

If the limit of vanishing nonlinearity is taken, then the centre of the conic section, since it

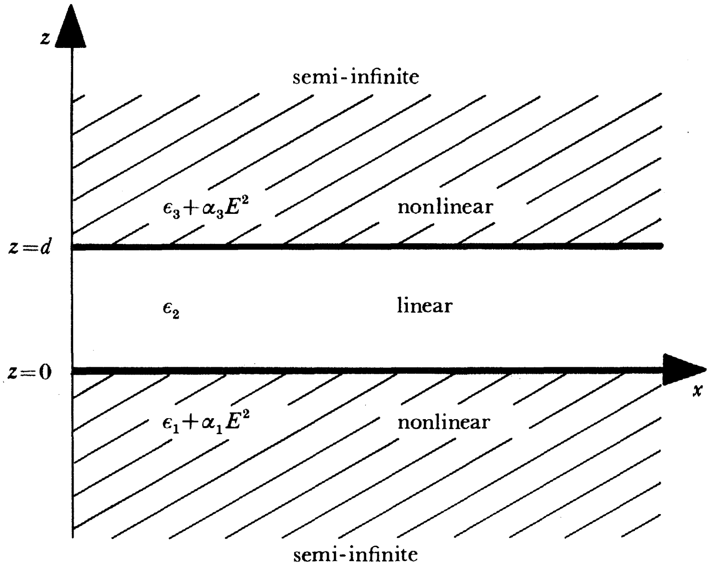

FIGURE 1. Asymmetric linear thin-film optical waveguide with nonlinear bounding dielectrics.

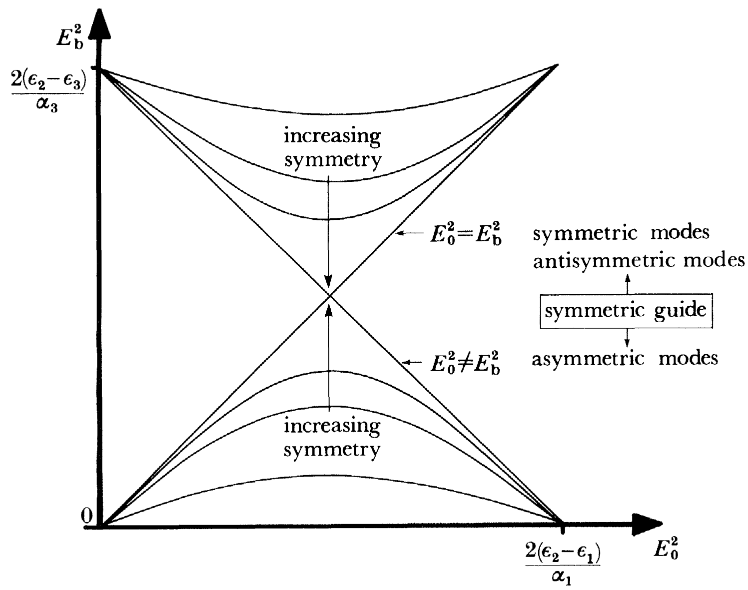

FIGURE 2. Type II boundary-field amplitude relation. In the limit of a symmetrically loaded film the hyperbolae degenerate to straight lines. Note that each set of hyperbolae has a different vertical scale.

has the coordinates $(\eta_1/\alpha_1, \eta_2/\alpha_2)$, moves out diagonally to infinity and the hyperbolae fan out into a set of straight lines of slope $(\epsilon_2-\epsilon_3)/(\epsilon_2-\epsilon_1)$. These lines are contained below $E_0^2 = E_b^2$ when $\epsilon_3 > \epsilon_1$. The significant point here is that asymmetric modes for which $E_0^2 \neq E_b^2$ occur both in the all linear structure and the nonlinear structure. This is not true of the symmetrically loaded slab. The limits on the E_b^2, E_0^2 axes show the containment within the guided-wave region.

TE MODE EIGENVALUE EQUATION

For a TE guided mode propagating along the x-axis of figure 1 with angular frequency ω and wavenumber k_x the electric field components in the media are given by:

$$\left.\begin{aligned}&\ddot{E}_2+k_2^2 E_2 = 0, \quad \ddot{E}_i-(k_i^2-2A_i E_i^2)\,E_i = 0,\\ &k_2^2 = \frac{\omega^2}{c^2}\epsilon_2-k_x^2, \quad A_i = \frac{\omega^2\alpha_i}{2c^2}, \quad k_i^2 = k_x^2-\frac{\omega^2}{c^2}\epsilon_i = \frac{\omega^2}{c^2}(\epsilon_2-\epsilon_i)-k_2^2, \quad i = 1, 3,\end{aligned}\right\} \tag{2}$$

where a dot over a variable denotes differentiation with respect to z, and c is the velocity of light in vacuo. It is also useful, at this stage, to define the quantities $\gamma_1^2 = k_1^2-A_1 E_0^2$, $\gamma_3^2 = k_3^2-A_3 E_b^2$ so that the gradient of the fields at $z = 0$ and $z = d$ can be written as $(\dot{E}_1)_{z=0} = \gamma_1 E_0$ and $(\dot{E}_3)_{z=d} = \gamma_3 E_b$, where γ_1 and γ_3 can be positive or negative.

After the application of the boundary conditions the eigenvalue equation reduces to

$$\cos(k_2 d) = \pm\left[\frac{\dfrac{c^2k_2^2}{\omega^2}\pm\left(\epsilon_2-\epsilon_1-\tfrac{1}{2}\alpha_1 E_0^2-\dfrac{c^2k_2^2}{\omega^2}\right)^{\frac{1}{2}}\left(\epsilon_2-\epsilon_3-\tfrac{1}{2}\alpha_3 E_b^2-\dfrac{c^2k_2^2}{\omega^2}\right)^{\frac{1}{2}}}{(\epsilon_2-\epsilon_1-\frac{1}{2}\alpha_1 E_0^2)^{\frac{1}{2}}(\epsilon_2-\epsilon_3-\frac{1}{2}\alpha_3 E_b^2)^{\frac{1}{2}}}\right]. \tag{3}$$

If the outer sign is taken as + in equation (3) then even parity solutions, for which E_0 and E_b have the same sign, yield asymmetric modes that, in the symmetric structure, give an even mode and an associated asymmetric mode. Now the slopes of the fields at each boundary determine whether the field immediately decays exponentially in the nonlinear medium or first rises to a maximum and then decays exponentially (i.e. whether a bulge appears). It is possible to distinguish various possibilities for E_3 and E_1 without actually needing to know their explicit form. Since the gradients of the fields at the slab boundaries are continuous then use of the field in the linear layer gives

$$\frac{k_2}{\sin(k_2 d)}[E_b-E_0\cos(k_2 d)] = \gamma_1 E_0, \quad \frac{k_2}{\sin(k_2 d)}[E_b\cos(k_2 d)-E_0] = \gamma_3 E_b. \tag{4}$$

Hence if $\sin(k_2 d) > 0$, $E_0 > 0$, $E_b > 0$ and $E_b/E_0 > \sec(k_2 d)$ then $\gamma_1 > 0$, $\gamma_3 > 0$ so that $E_1 = \gamma_1 E_0$ and $E_3 = \gamma_3 E_b$. This shows, immediately, that the field in the lower medium decays exponentially while the field in the upper nonlinear medium possesses a maximum (i.e. a bulge). On the other hand, if $0 < E_b/E_0 < \cos(k_2 d)$ then $\gamma_1 < 0$, $\gamma_3 < 0$ and $E_1 = -|\gamma_1|\,E_0$, $E_3 = -|\gamma_3|\,E_b$, which shows that the lower nonlinear medium now possesses a bulge while the field in the upper medium decays exponentially. The intermediate case

$$\cos(k_2 d) < E_b/E_0 < \sec(k_2 d) \quad \text{gives} \quad \gamma_1 > 0, \gamma_3 < 0$$

and the fields in either nonlinear medium decay exponentially without bulges. All of these arguments refer to the range $0 < k_2 d < \frac{1}{2}\pi$. In the other ranges possibilities exist for bulges to appear in both the upper and lower media. It will depend upon the thickness of the layer and the other parameters as to whether these are TE_0, TE_1, ... mode ranges.

Figure 3 shows the dependence of ck_x/ω, for some specimen data, upon $\frac{1}{2}\alpha_1 E_0^2$. The upper and lower branches of figure 2 are analysed separately, but each shows a cut-off locus that depends upon E_0.

If $E_0 > 0$, $E_b > 0$, the first allowed range of k_2, for which the outer sign is positive and the inner sign is negative in (3), ends when $\gamma_1 = 0$ at

$$\frac{k_2}{\omega(\epsilon_2 - \epsilon_3 - \frac{1}{2}\alpha_3 E_b^2)^{\frac{1}{2}}/c} = \cos(k_2 d) = \pm\frac{E_b}{E_0}. \quad (5)$$

This is a transition from $\gamma_1 > 0$ to $\gamma_1 < 0$ so, since the slope of the field at the boundary in medium 1 is $E_1 = \gamma_1 E_0$, no calculation is needed to understand that this is possible only if a maximum (i.e. a *single bulge*) occurs in the electric field of the nonlinear medium. A further interesting phenomenon occurs, however. This transition point is also where k_2 begins to satisfy (3) with positive inner sign. Thus the transition point is a cusp and allowed eigenvalues do not cease to exist as the single-bulge range is entered.

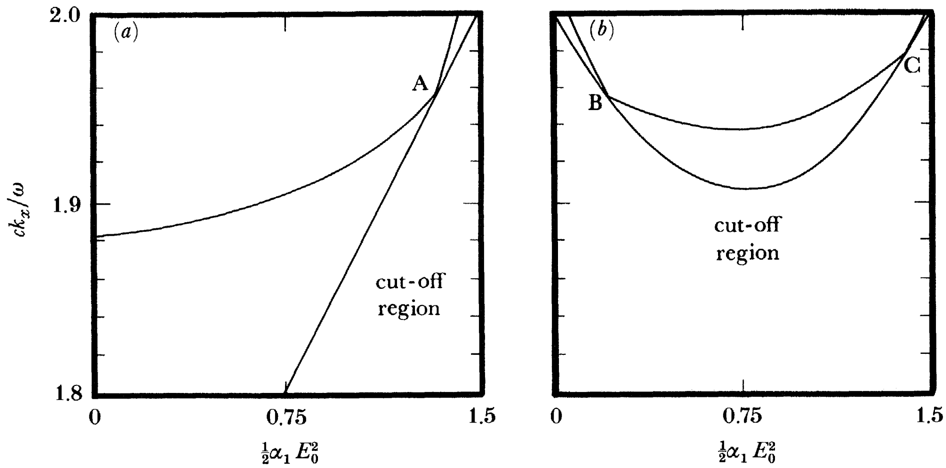

FIGURE 3. Dependence of guided wavenumber on the optical nonlinearity. Data: $\epsilon_1 = 2.5$, $\epsilon_2 = 4.0$, $\epsilon_3 = 2.0$, $d = 0.5\lambda$, $\lambda = 2\pi c/\omega$, $\beta = 1.4$; (*a*) corresponds to a lower hyperbola of figure 2; (*b*) corresponds to an upper hyperbola. A indicates the point of the appearance of maximum field in the lower medium and B, C, in the upper medium.

For the upper curve in the (E_0^2, E_b^2) plot, E_b^2 is, initially, at a finite value and is substantially larger than E_0^2 for all E_0. Cut off will, therefore, occur at $c^2k_2^2/\omega^2 = \epsilon_2 - \epsilon_1 - \frac{1}{2}\alpha_1 E_0^2$ or $\epsilon_2 - \epsilon_3 - \frac{1}{2}\alpha_3 E_b^2$, whichever is the smaller. This leads to a very curved cut-off locus involving both E_0 and E_b.

Optical hysteresis

The possibility that the thin film structure can exhibit hysteresis and possible optical bistability is determined from the behaviour of the power flow down the guide. This power flow can be calculated without knowing the fields in the outside nonlinear media, a feature that was seen above to be also true of the dispersion equation. Indeed, up to now, the fields in the bounding media have not been used at all.

For the outside bounding media

$$(\dot{E}_i/E_i)^2 = k_i^2 - A_i E_i^2. \quad (6)$$

Differentiation with respect to z gives

$$2\left(\frac{\dot{E}_i}{E_i}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\dot{E}_i}{E_i}\right) = -2A_i E_i \dot{E}_i, \quad \int_a^b E_i^2\,\mathrm{d}z = -\frac{1}{A_i}\left(\frac{\dot{E}_i}{E_i}\right)_a^b. \quad (7)$$

For $i = 1$, $b = 0$, $a = -\infty$ and for $i = 3$, $b = \infty$, $a = d$ where

$$\lim_{z\to-\infty}\left(\frac{\dot{E}_1}{E_1}\right) = k_1, \quad \lim_{z\to\infty}\left(\frac{\dot{E}_3}{E_3}\right) = -k_3$$

and, for any other value of z, $\dot{E}_i/E_i$ is $(k_i^2 - A_i E_i^2)^{\frac{1}{2}}$. The power flow integrals in media 1 and 3 therefore evaluate to $(k_{1,3} \pm |\gamma_{1,3}|)/A_{1,3}$ with the restrictions that $\dot{E}_3/E_3 > 0$, $\dot{E}_1/E_1 < 0$ for the positive sign and $\dot{E}_1/E_1 > 0$, $\dot{E}_3/E_3 < 0$ for the negative sign at the boundaries.

In the linear thin film

$$E_2^2 = \frac{1}{2k_2^2}\left(c_2 - \frac{\mathrm{d}}{\mathrm{d}z}(E_2\dot{E}_2)\right), \tag{8}$$

so that the power integral becomes

$$\int_0^d E_2^2\,\mathrm{d}z = \frac{1}{2k_2^2}[c_2 d - (E_2\dot{E}_2)_d + (E_2\dot{E}_2)_0], \tag{9}$$

but

$$(\dot{E}_2)_d = \gamma_3 E_\mathrm{b}, \quad (\dot{E}_1)_0 = \gamma_1 E_0, \quad c_2 = (k_2^2+\gamma_1^2)\,E_0^2 = (k_2^2+\gamma_3^2)\,E_\mathrm{b}^2, \tag{10}$$

so that the total power flow in the thin film and the nonlinear bounding media is

$$P = P_0 K_x\{\beta(K_1 \pm G_1) + K_3 \mp G_3 + [D(\epsilon_2 - \epsilon_1 - \tfrac{1}{2}\alpha_1 E_0^2)\,\tfrac{1}{2}\beta\alpha_1 E_0^2 \mp \tfrac{1}{2}G_1\beta\alpha_1 E_0^2 \pm \tfrac{1}{2}G_3\alpha_3 E_\mathrm{b}^2]/2K_2^2\}, \tag{11}$$

where $K_x = ck_x/\omega$, $K_{1,3} = ck_{1,3}/\omega$, $G_{1,3} = c\gamma_{1,3}/\omega$, $\beta = \alpha_3/\alpha_1$, $D = cd/\omega$ and $P_0 = 2c^2\epsilon_0/\alpha_3\omega$ has dimensions of watts per metre. Since E_b^2 is expressible in terms of E_0^2 and, through the dispersion equation, E_0^2 is a function of K_x, then P/P_0 can be varied with K_x for a particular value of ω.

Figure 4 shows the power flow variation with K_x for dielectric constants that are fairly close together. This is a TE_0 mode and shows two distinct power loops. If the system were to become symmetric these loops would merge into a symmetric and an associated asymmetric mode. These curves only seem to be like the symmetric case. In the latter structure it is the upper parts of the upper loop that would be responsible for any bistability. In fact, even a small amount of asymmetry is sufficient to split the curves and thus lift the degeneracy of symmetric and asymmetric modes into two distinct asymmetric modes. If $E_b/E_0 < 0$ a strong peak develops at much higher power and only if $d > 2\lambda$.

The power curves can now be used to predict any optical hysteresis. Suppose that the nonlinear media are lossy (an assumption that the media are infinitely lossy will suffice for the present purposes), then massive energy dumping through the appearance of electric field bulges can be anticipated and the variation of P_out, the power out of the guide, with P_in, the input power to the guide, can be calculated. In the first place, the power shown in figure 4 is the input power distributed over the whole of a plane perpendicular to the z axis. This energy is, as a consequence, distributed over both the linear layer and the nonlinear media. If at some distance down the guide no absorption has occurred then the power out through a similar plane must be exactly equal to the input power.

As P_in advances up the lower loop of figure 4 the bulk of the power flow is in the linear layer so that P_out remains essentially the same as P_in. As the maximum is approached deviation from linearity occurs as the field strength in the nonlinear media grows. At the maximum in

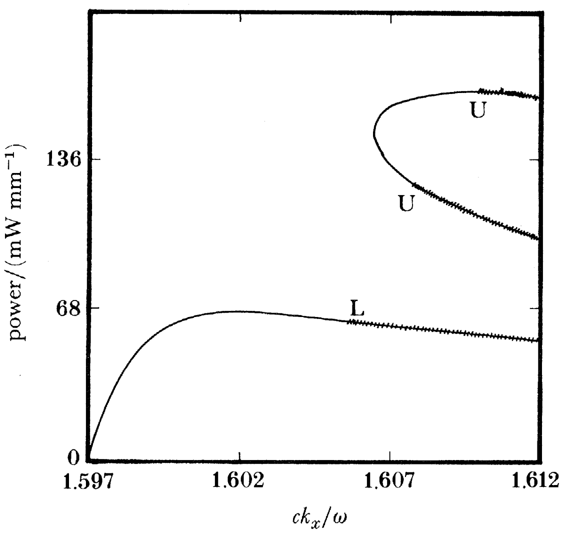

FIGURE 4. Total power flow down the waveguide as a function of ck_x/ω. The hatched part of the curves marks the appearance of a sustained field maximum (bulge) in the upper (U) or lower (L) nonlinear medium. Data: $\epsilon_1 = 2.45$, $\epsilon_2 = 2.6$, $\epsilon_3 = 2.3$, $d = 1.5\lambda$, $\alpha_3 = 6.37 \times 10^{-12}$ m^2 V^{-2}, $\beta = 1.05$, $\lambda = 0.515$ μm.

the power curve a further increase in P_{in} causes no further increase in P_{out}, i.e. the power saturates. A much greater increase in P_{in} can cause a transition to the higher power loop.

A reduction of P_{in} from the maximum opens up the option of returning down the first branch, the second branch on the large K_x side of the peak, or both branches. Suppose that to set a limit on the hysteresis curve a descent down the far side of the peak occurs so that more and more energy is dumped into the nonlinear medium as the wavenumber at which a bulge in the field of the lower nonlinear medium is approached. This causes P_{out} to fall below the P_{out} obtained on the outward direction, as P_{in} is reduced. This fall away continues as higher K_x values are accessed until a point is reached where a transition back to a guided wave on the outward branch becomes possible, or a surface mode is generated. It is, of course, possible that a transition to the outward branch could have occurred earlier, but the hysteresis loop shown in figure 5 marks the limit of the possibilities. Note that a hysteresis loop can also be derived for the upper power loop of figure 4 and that the data used for figure 3 would give a highly structured form of hysteresis.

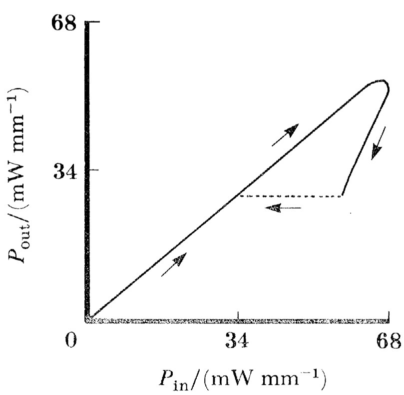

FIGURE 5. Optical hysteresis shown in the output power (P_{out}) against input power (P_{in}) characteristic obtained from figure 4.

Hysteresis has recently been demonstrated, for TE_1 modes, experimentally by Vach *et al.* (1984), for a nonlinear medium bounded by a linear film on a linear substrate. This paper shows that many other possibilities occur in the type of asymmetric structure discussed above. Guided waves offer the prospect of non-resonant bistable devices, but it should be emphasized that a lot of theoretical work needs to be done before the stability of these transformations and the genuine prospects for bistable operation can be determined.

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